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LETTER TO THE EDITOR

Logarithmic corrections to finite-size scaling in strips

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Abstract. The corrections to the finite-size scaling behaviour of the eigenvalues of the transfer matrix of a critical theory defined on an infinitely long strip of finite width, which occur when the Hamiltonian contains a marginal operator, are computed using conformal invariance. They show a calculable universal logarthmic character. For the four-state Potts model they agree with numerical data.

One of the most immediate applications of conformal invariance in two-dimensional critical behaviour results from the observation (Cardy 1984) that, if we denote the transfer matrix of a strip of width L, with periodic boundary conditions, by $\exp(-\hat{H})$, then the eigenvalues E_n of $\hat{H}$ are related to the scaling dimensions x_n of the scaling operators of the theory by

$$E_n - E_0 \sim 2\pi x_n / L \tag{1}$$

in the limit $L \to \infty$. In addition (Blöte *et al* 1986, Affleck 1986), the ground state energy E_0, which is the free energy per unit length, is related to the conformal anomaly number c, which plays an important role in the formal classification scheme (Friedan *et al* 1984), by

$$E_0 = AL - \pi c/6L + o(L^{-1}) \tag{2}$$

where A is the bulk free energy per unit area, a non-universal constant.

By now these relations have been verified for a large number of models and they form a powerful means of investigating new ones. The corrections to (1) due to the presence of irrelevant operators have already been considered (Cardy 1986a, b). As expected, they are of order $L^{-1-|y|}$, where y is the renormalisation group eigenvalue of the irrelevant operator. Similar corrections are to be expected in (2). However, in the presence of a marginally irrelevant operator, one typically expects logarithmic terms. In this letter, we calculate the leading logarithmic corrections to the results in (1) and (2) and show that they have a universal form. Our main results are contained in (11) and (22).

To be specific, let us consider the effect of adding a term $(-g)\,\Sigma_r\,\phi(r)$ to the fixed-point Hamiltonian. The non-linear scaling field g is supposed to have scaling dimension $x = 2 - y$, where y is its renormalisation group eigenvalue. Under a change of length scale it satisfies the renormalisation group equation

$$dg/dl = (2-x)g - \pi b g^2 + O(g^3) \tag{3}$$

where $x=2$ and $b>0$ if g is marginally irrelevant. In this case the solution for large l is

$$g(l)=\frac{g}{1+\pi bgl}. \tag{4}$$

In addition, the other scaling variables u_n are supposed to satisfy the renormalisation group equations

$$\mathrm{d}u_n/\mathrm{d}l=(2-x_n)u_n-2\pi b_n g u_n+\mathrm{O}(g^2u_n). \tag{5}$$

The numbers b and b_n are universal if ϕ is normalised so that its two-point function is $|r|^{-2x}$. In fact, they are related to operator product expansion coefficients or, equivalently, they give the normalisation of the three-point functions

$$\langle\phi(r_1)\phi(r_2)\phi(r_3)\rangle=-b/|r_{12}|^x|r_{23}|^x|r_{31}|^x \tag{6}$$

and

$$\langle\phi(r_1)\phi_n(r_2)\phi_n(r_3)\rangle=-b_n/|r_{12}|^x|r_{13}|^x|r_{23}|^{2x_n-x}. \tag{7}$$

The easiest way to see this is to perform a Kosterlitz-type renormalisation group calculation (Kosterlitz 1974) on the expansion of the free energy in the infinite system. An example of such a calculation for the random Potts model is given in Ludwig (1986).

We discuss the form of the corrections to (1) as predicted by the renormalisation group. Since the left-hand side is an inverse correlation length, under a rescaling it transforms according to

$$\xi^{-1}(g,L^{-1})=\mathrm{e}^{-l}\xi^{-1}(g(l),L^{-1}\,\mathrm{e}^{l}). \tag{8}$$

Choosing $\mathrm{e}^l=L$ and using (4), we obtain the scaling prediction

$$E_n-E_0\sim L^{-1}\Phi_n\left(\frac{g}{1+\pi bg\ln L}\right) \tag{9}$$

where Φ_n is a universal function. In Cardy (1986a, b) we showed that the O(g) correction to (1), for arbitrary x, is given by

$$E_n-E_0\sim 2\pi x_n/L+gb_nL(2\pi/L)^x+\mathrm{O}(g^2). \tag{10}$$

Comparing (9) and (10), we see that, in the limit $L\to\infty$, the corrections to (1) are independent of the value of g and are of the universal form

$$E_n-E_0\sim(2\pi/L)(x_n+d_n/\ln L)+\mathrm{O}(L^{-1}(\ln L)^{-2}) \tag{11}$$

where $d_n=2b_n/b$. It should be stressed that this behaviour will set in for strip widths $L\gg\exp(1/bg)$, which may be very large if g is small. An interesting special case is when $\phi_n=\phi$, corresponding to an energy gap scaling asymptotically like $4\pi/L$. In that case the logarithmic correction is as in (11) with $x_n=d_n=2$.

Next we discuss the corrections to E_0. It is convenient to consider the free energy per unit area $f=E_0/L$. The change δf in this quantity may be calculated in an expansion in g involving the correlation functions of ϕ evaluated at the fixed point:

$$\delta f=-g\langle\phi\rangle-\frac{g^2}{2!}\sum_r\langle\phi(r)\phi(0)\rangle_c-\frac{g^3}{3!}\sum_{r_1,r_2}\langle\phi(r_1)\phi(r_2)\phi(0)\rangle_c+\dots. \tag{12}$$

In general, operators may be subtracted so that $\langle\phi\rangle=0$ in the bulk, and conformal invariance then implies that the first term vanishes in the strip also. However, operators in the conformal block of the unit operator transform anomalously (Belavin *et al* 1984) and will have a non-zero expectation value in the strip. The most important is the irrelevant operator $L_{-2}\bar{L}_{-2}\mathbf{1}$, which has $x=4$. This will give $\mathrm{O}(L^{-4})$ corrections to f which may be expected to have a large amplitude, since they occur in first order. Indeed, such corrections are observed to be important in numerical calculations (Blöte and Nightingale 1982). They may mask the more interesting corrections we are studying here. In the higher-order terms, the sums over r may be replaced by integrals, and the correlation functions by their continuum limits, if we impose a short distance cutoff $r>1$, etc. (Here, as throughout this letter, we take the lattice spacing to give the unit of length.) The correlation functions are to be calculated in the strip geometry and are determined in terms of the infinite plane correlation functions by using the conformal mapping (Cardy 1984) $w=(L/2\pi)\ln z$. However it is simpler to transform the integrals back to the z plane. We first examine the $\mathrm{O}(g^2)$ term. This is, for general x,

$$\delta f_2 = -\tfrac{1}{2}g^2(2\pi/L)^{2x-2}I_2(x, 2\pi/L) \tag{13}$$

where

$$I_2(x,\varepsilon) = \int \mathrm{d}^2z\,|z^{x-2}|z-1|^{-x}\theta(|z-1|-\varepsilon). \tag{14}$$

For $0<x<1$, $I_2(x,0)$ converges and is equal to (Hentschke *et al* 1986)

$$\frac{\pi\Gamma(x/2)^2\Gamma(1-x)}{\Gamma(1-x/2)^2\Gamma(x)}. \tag{15}$$

The leading $\varepsilon\to 0$ behaviour of $I_2(x,\varepsilon)$ is easily extracted. We then find that

$$I_2(x,\varepsilon) = I_2(x,0) - \frac{\pi\varepsilon^{2-2x}}{1-x} + \mathrm{O}(\varepsilon^{6-2x}). \tag{16}$$

Note that this is valid up to the next pole of $I_2(x,0)$, which occurs at $x=3$. The second term in (16) gives a contribution to the bulk free energy. It is non-universal because it depends on the details of the cutoff procedure. The first term contributes to the universal L^{-2} term. If g is irrelevant ($x>2$), we see that the leading correction to the free energy has the form

$$f = A(g) - \frac{\pi}{6L^2}(c + Bg^2L^{4-2x} + \mathrm{O}(L^{6-3x})) \tag{17}$$

where B is a constant which is positive for $2<x<3$. The expression in parentheses defines an effective $c(L)$, which approaches its asymptotic value from above.

However, from (15) we see that this correction actually vanishes at $x=2$. We therefore must consider the next term. This is

$$\delta f_3 = \tfrac{1}{6}bg^3(2\pi/L)^{3x-4}I_3(x, 2\pi/L) \tag{18}$$

where

$$I_3(x,0) = \int |z_1|^{x-2}|z_2|^{x-2}|z_1-1|^{-x}|z_2-1|^{-x}|z_1-z_2|^{-x}\,\mathrm{d}^2z_1\,\mathrm{d}^2z_2. \tag{19}$$

This converges for $0<x<\frac{4}{3}$. The pole at $x=\frac{4}{3}$ corresponds, by the same argument as above, to a contribution to the bulk energy. The term we want is given by the analytic continuation of $I_3(x,0)$ to $x=2$. An integration by parts shows that

$$I_3(x,0) = \frac{2-x}{4-3x}\int F(z_1,z_2)(|z_1-1|^2+|z_1|^2-1)|z_1|^{-2}\,\mathrm{d}^2z_1\,\mathrm{d}^2z_2 \tag{20}$$

where F is the same integrand as in (19). The integral now converges for $0<x<2$ and it is relatively straightforward to determine the residue of the pole at $x=2$. The final result is $I_3(2,0)=-\pi^2$, so that

$$\delta f=-2\pi^4 bg^3/3L^2+O(g^4). \tag{21}$$

The renormalisation group (Blöte and Nightingale 1982) now implies that g in the above expression should be replaced by $g(\ln L)\sim 1/\pi b \ln L$. This, of course, means that we should pick up logarithmic dependences on ε at higher orders in g. Although conformal invariance does not in general completely predict the four- and higher-point functions, one can see, by using the operator product expansion, from where these factors must arise. Note that no such factors arise which would correspond to replacing g by $g(\ln L)$ in the non-universal bulk term, in agreement with Blöte and Nightingale (1982). The final result for the free energy is

$$f=E_0/L=A(g)-\frac{\pi}{6L^2}\left(c+\frac{4}{b^2}(\ln L)^{-3}+O((\ln L)^{-4})\right). \tag{22}$$

Note that the effective $c(L)$ defined by (22) approaches its asymptotic value from above.

For the case of the four-state Potts model, we have obtained the operator product expansion coefficients $-b$ and $-b_\varepsilon$, where ε is the energy density, by taking the limit $q\to 4$ of the results of Dotsenko and Fateev (1985). We find $b=4/\sqrt{3}$ and $b_\varepsilon=\sqrt{3}/2$. The ratio of these, as they appear in the renormalisation group equations (3) and (5), agrees with the ratio of coefficients found by Cardy *et al* (1980) (who used a different normalisation of the marginal scaling field g) by comparison with exact results. This result forms a non-trivial check of the work of Dotsenko and Fateev. We thus see that $d_\varepsilon=\frac{3}{4}$. In a similar way, we find that $d_\sigma=\frac{1}{16}$, where σ is the leading magnetisation operator.

Blöte and Nightingale (1982) and Nightingale and Blöte (1983) have calculated numerically the free energy per site f, and the gaps $E_\varepsilon-E_0$ and $E_\sigma-E_0$, corresponding to the leading energy and magnetisation operators, repectively, for the nearest-neighbour isotropic $q=4$ Potts model for strips up to width $L_{\max}\leq 11$. As discussed in Nightingale and Blöte (1983) and Blöte *et al* (1986), the extrapolated values of x_ε, x_σ and c agree fairly well with their expected exact values of $\frac{1}{2}$, $\frac{1}{8}$ and 1, respectively, although the agreement is not so good as for other values of q, where no logarithmic corrections are present. We now show that if these exact values are taken for granted, then the finite-size deviations found by Blöte and Nightingale are accounted for by logarithmic corrections of the type discussed above. In calculating these it is not appropriate to use the asymptotic forms in (11) and (22), but rather to replace g in (10) and (21) by $g(\ln L)$. We used the free energy data to calculate $g(\ln L)$ and then used this value to predict the deviations in the other eigenvalues. Although the correction δf is typically down by a factor of 10^{-4} on the leading behaviour, the results of Blöte and Nightingale are sufficiently precise to allow us to extract these values easily. Our results for the deviations in the gaps are shown in table 1 and compared with the same quantities derived from the data displayed in Nightingale and Blöte (1983). The agreement is rather good, despite the fact that that we have ignored corrections of higher order in $g(\ln L)$. Consideration of these terms shows that the effective expansion parameter is $\pi g(\ln L)\approx 0.1$, so that the discrepancies are of the expected order of magnitude. A plot of $g(\ln L)^{-1}$ against $\ln L$ shows some curvature, indicating that the asymptotic form of (4) has not yet been reached. The observed

Table 1. Corrections to the free energy per site and the lowest gaps of the four-state Potts model, for strips of width L. Second column shows $\delta f = f(L) - f(\infty) + \pi/6L^2$, with $f(L)$ taken from Blöte and Nightingale (1982) and $f(\infty)$ from Baxter (1973). The last four columns show predicted corrections to the gaps: $\Delta_n = (E_n(L) - E_0(L))L - 2\pi x_n$, and their exact values, derived from data which is displayed in figures 2(a) and (b) of Nightingale and Blöte (1983).

L	$\delta f \times 10^5$	$g(\ln L)$	Δ_ε	Exact	Δ_σ	Exact
6	42.17	0.0466	1.59	1.67	0.132	0.114
7	23.65	0.0426	1.46	1.60	0.121	0.105
8	14.53	0.0396	1.35	1.54	0.112	0.100
9	9.55	0.0372	1.27	1.49	0.106	—
10	6.60	0.0353	1.21	1.45	0.101	—

departures may be accounted for by adding a term $O(g^3)$, with an appropriate cofficient, to the right-hand side of (3).

We conclude that the major corrections to finite-size scaling for this model are logarithmic and that their amplitudes show quantitative agreement with the predictions of conformal invariance.

More recently Alcaraz and Barber (1987) have considered the quantum Hamiltonian four-state Potts model and another Ising model which is expected to be in the same universality class. For the latter model, they find effective values of c less than unity, which would appear to contradict our result, or to imply that it is in a different universality class. However, the situation in this case is complicated by the anisotropy, which is affected in a non-trivial way by corrections to scaling. In the estimates of the ratios c/x_σ, where the anisotropy should cancel, for both models the sign of the deviations from the exact Potts values found by Alcaraz and Barber agrees with our results, although the L dependence does not. It is possible that the asymptotic region begins at larger values of L for these quantum models. Similar poor convergence for the $q = 4$ quantum chain was found by von Gehlen *et al* (1986).

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